JAG Winter Camp '08
Day 2 Problem E
Subdividing a Land

ACM-ICPC
Japanese Alumni Group

About the Problem

- Original problem by T. Yoshino

- Sample Solution by T. Yoshino, Y. Hirano
- Problem Statement by T. Yoshino

- Presentation by T. Yoshino

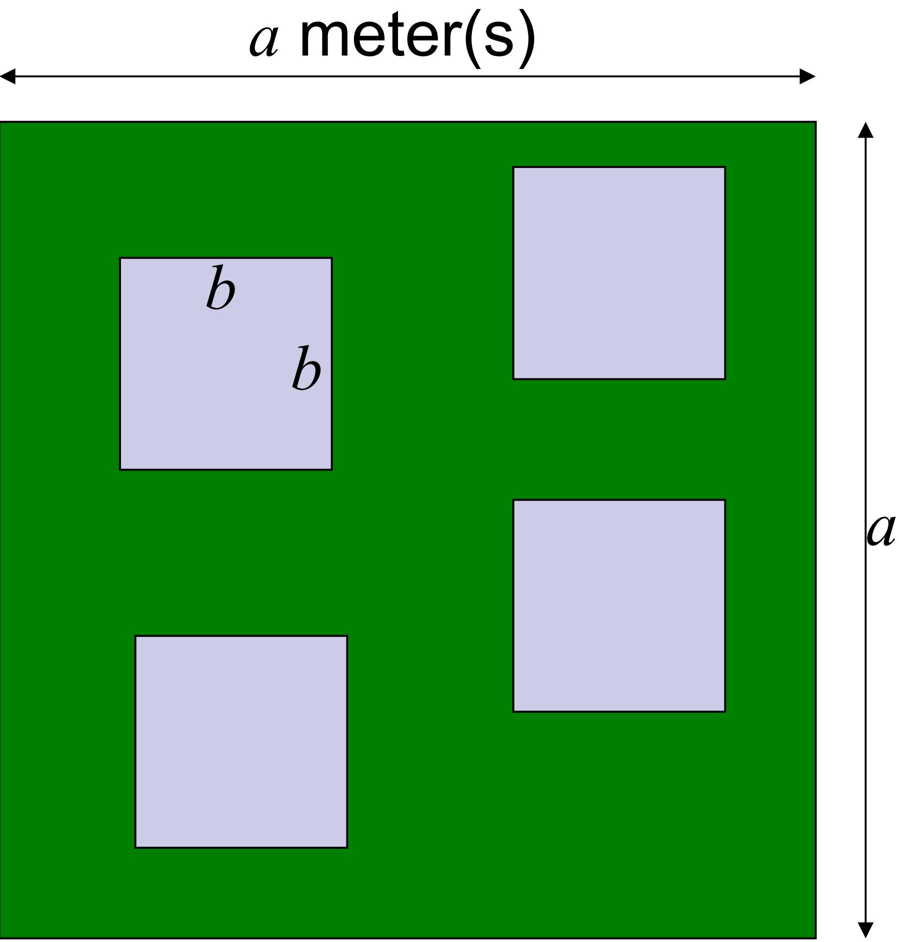
Contest Result

- 14 submits during the contest
- Only 1 team was accepted
  - At 234 min. (____________)
    - This is the corrected result, as we had a ‘%lld’ problem

Overview of the Problem

- From a a-meter square, take n b-meter squares
  - a, b, n are integers
- But the total ratio must not exceed 50%
  - The residential area less than 50% becomes *dead space*
- Minimize the dead space!
  - Not by percentage, but area
    - If we calculate dead space by percentage, it can be as smaller as we want simply by increasing the denominator
    - This leads to divergence to infinity !!

Overview of the Problem



- Total area of must not exceed 50%
  - More than or equal to 50% of must be retained
- Minimize dead space
  - Dead space = exceeding 50%
  - $= (a^2 - nb^2) - 1/2 \cdot a^2$

Solution

- From the percentage constraint,

$$a^2 - 2nb^2 \geq 0$$

  - And want to minimize: $1/2 \cdot a^2 - nb^2 = \frac{1}{2}(a^2 - 2nb^2)$
- Actually there are two cases to consider:

1. $a^2 - 2nb^2 = 0$
2. $a^2 - 2nb^2 = 1$

  - These two equations cover all cases, so no more cases to be concerned

This is Not a Packing Problem

- Thanks to the 50% limit, we can always take n squares
  - From the constraint, $a/b \geq \sqrt{2n}$
  - Apparently, a-meter square can contain $\lfloor a/b \rfloor^2$ b-meter squares
  - **Theorem 1.** For all positive integer n, $\lfloor \sqrt{2n} \rfloor^2 \geq n$.

    Proof. This is easy to prove. $\square$

Case 1:
$$a^2 - 2nb^2 = 0$$

- The equation can be factorized into:

$$\left(a + \sqrt{2n}b\right)\left(a - \sqrt{2n}b\right) = 0$$

  - This equation has integer solutions only if $2n$ is a perfect square of a certain integer
  - Let m be this integer, the minimum solution is apparently $(a, b) = (m, 1)$.

Case 2:

$$a^2 - 2nb^2 = 1$$

- This is the equation well-known as *"Pell's equation"* :

$$x^2 - Ny^2 = 1$$

  - This equation always has (infinitely many) answers whenever N is not a perfect square (Lagrange, ca. 1766)
    - So these two cases cover all inputs

Never Perform a Naïve Search!

- Sometimes solutions of the equation becomes very huge
  - For example, the minimum solution of
$$x^2 - 61y^2 = 1$$
is $(x, y) = (1766319049, 226153980)$
  - As stated in the problem, the solution of each test case is less than 2^{63}
    - Actually, you have to totally inspect 8.92×10^{18} cases for this problem!!

So How to Solve the Equation?

- Pell's equation can be solved by calculating the convergent of continued fraction of $\sqrt{N}$
  - Intuitively, this is to calculate a fraction x / y that approximates $\sqrt{N}$
  - The solution is the case where the fraction approaches to $\sqrt{N}$ sufficiently

Continued Fraction

- A real number ω can be represented by a positive integer sequence $[a_0; a_1, a_2, \ldots]$ where

$$\omega = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

  - An irrational number has (only one) infinite continued fraction representation
    - A rational number has a finite sequence

Calculating the Convergent

- The next algorithm illustrates how to calculate the convergent $[a_0; a_1, \ldots]$ (of $\sqrt{N}$)
  - First, let $\omega_0 = \sqrt{N}$
  - Let $a_n = \lfloor \omega_n \rfloor$, the largest integer less than or equal to ω_n
  - Calculate ω_{n+1} by the next equation:

$$\omega_{n+1} = \frac{1}{\omega_n - a_n}$$

  - Repeat these steps

From a Continued Fraction To a Rational Number

- Suppose we take the first $(n+1)$ elements of the convergent
  - Let the elements be $[a_0; a_1, \ldots, a_n]$
  - Then, the sequence expresses a rational number p_{n+1} / q_{n+1} where

$$\begin{cases} p_{-1} = 0,\ q_{-1} = 1 \\ p_0 = 1,\ q_0 = 0 \\ p_{n+1} = p_{n-1} + a_n p_n,\ q_{n+1} = q_{n-1} + a_n q_n \quad (n \geq 0) \end{cases}$$

Some Theorems About the Solution of Pell's Equation

- There are some theorems regarding the existence of solutions of the equation
  - **Theorem 2** (Lagrange). There exists an integer $n > 0$ such that
    $$p_n{}^2 - Nq_n{}^2 = 1$$
  - **Theorem 3** (Euler). Among such n's, the least one gives the minimum nontrivial solution to the Pell's equation.
  - So basically, we have just to calculate the sequences p_n and q_n until the solution is found

Normal Floating-Point Number Is Not Appropriate for This Purpose

- Naïve implementation of the former algorithm requires multiprecision number
  - Underflow error becomes a problem in the calculation of ω_{n+1}
    - Since ω_n - a_n becomes nearly 0
  - This can lead to infinite loop
    - One of teams tried this and got a Runtime Error
  - Simply avoid that!

Solution Techniques

- Actually, the following fact is known:

Theorem 4 (Euler, 1765). There exists integers g_n and h_n such that

$$\omega_n = \frac{g_n + \sqrt{N}}{h_n}$$

- By substituting this to the equation for ω_{n+1}, we get:

$$\begin{cases} g_0 = 0,\ h_0 = 1 \\ g_{n+1} = -g_n + a_n h_n,\ h_{n+1} = \dfrac{N - {g_{n+1}}^2}{h_n} \quad (n > 0) \end{cases}$$

  - And from Theorem 4, $a_n = \lfloor \dfrac{a_0 + g_n}{h_n} \rfloor$

How Much Precision We Need to Calculate?

- **Theorem 5.** $\forall n \geq 0.\ g_n \leq a_0 = \lfloor \sqrt{N} \rfloor$

  Proof. It is obvious from the fact $a_n h_n = \lfloor \frac{a_0 + g_n}{h_n} \rfloor h_n \leq a_0 + g_n$. □

- **Theorem 6.** $\forall n \geq 0.\ 0 < h_n \leq N$

  Proof. By induction. Obvious about the basis $n = 0$.
  Since $g_n < \sqrt{N}$, $h_{n+1} > 0$. And $h_{n+1} \leq N$ since ${g_{n+1}}^2 \geq 0$. □

- **Theorem 7.** $\forall n \geq 0.\ 0 < a_n \leq 2a_0$

  Proof. By induction. Basis is obvious.
  From the definition, $b_n - a_n < 1$. Then $a_{n+1} \geq 1 > 0$ follows.
  And by using Theorem 5 and 6, $a_{n+1} \leq a_0 + a_0 = 2a_0$. □

How Much Precision We Need to Calculate? (Cont'd)

- So g_n, h_n, a_n fit into `int`'s
  - No need to care for overflows

- p_n, q_n are obviously growing as n increases, so use `long long` for them
  - From the problem statement, it is guaranteed that the answer never exceeds 2^{63}

Speed Up Technique

- Actually, $h_n = |{p_n}^2 - N{q_n}^2|$ holds
  - So no need to calculate RHS of the equation every time
    - Or, this relation can be used to bound the range of RHS value of the equation
  - Just break a loop once h_n becomes 1
    - But be aware that h_n contains an absolute value
    - Post-processing needed when RHS=-1
      - (p', q') is the minimum answer for RHS=+1, where $p' = {p_n}^2 + N{q_n}^2,\ q' = 2p_nq_n$

Some Interesting Properties of Pell's Equation

- Consider a quadratic field

$$\mathbb{Q}(\sqrt{N}) = \left\{ x + y\sqrt{N} | x, y \in \mathbb{Q} \right\}$$

  - LHS of Pell's Equation is a norm on the field

$$||\alpha|| = \alpha \cdot \overline{\alpha} = (x + y\sqrt{N})(x - y\sqrt{N}) = x^2 - Ny^2$$

  - Let α_1 be the minimum solution, then for all integer $k \geqq 0$, α_1^k is also a solution
    - Actually, all solution of the Pell's equation can be represented as above

About Judge Data

- As you will expect, we have prepared the data which does exhaustive search
  - Total 6,921 cases
  - Maximum value of solution is at $n = 6621$
    - $(a, b) = (8987289718054858751, 7810016479202720)$
    - These are less than $2^{63} = 9223372036854775808$

Judges’ Solution

- By T. Yoshino
  - 54 lines in C++ (removed comments / unused code)
    - The core loop can be written in ~10 lines.
  - Solves the judge data in <1 sec
- By Y. Hirano
  - 132 lines in C++
  - Naïve calculation of a convergent on a field $\mathbb{Q}(\sqrt{N})$
  - Solves the judge data in about 3 sec.
- Both will do

References

- Pell’s Equation in Wikipedia
  - History and Solution Techniques
  - Also see Continued Fraction
- Pell Equation in Wolfram MathWorld

Any Questions?